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A note on the stability of a family of space-periodic Beltrami flows

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The linear stability of the 'ABC' flows $\boldsymbol{u} = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x)$ is investigated numerically, in the presence of dissipation, for the case where the perturbation has the same 2π-periodicity as the basic flow. Above a critical Reynolds number, the flows are in general found to be unstable, with a growth time that becomes comparable to the dynamical timescale of the flow as the Reynolds number becomes large. The fastest-growing disturbance field is spatially intermittent, and reaches its peak intensity in features which are localized within or at the edge of regions where the undisturbed flow is chaotic, as occurs in the corresponding MHD problem.

1. Introduction

Recently there has been a spate of interest in the family of flows whose periodic velocity components are given by

$$\left.\begin{aligned} u &= A \sin z + C \cos y, \\ v &= B \sin x + A \cos z, \\ w &= C \sin y + B \cos x, \end{aligned}\right\} \tag{1.1}$$

first introduced by Arnol'd (1965). These are Beltrami flows satisfying $\nabla \wedge \boldsymbol{u} = \boldsymbol{u}$, and the case with $A = B = C$ was utilized by Childress (1970) in connection with the kinematic dynamo problem. They have thus been referred to as 'ABC flows' by Dombre *et al.* (1986), who conducted an extensive study into the dynamical properties of (1.1). As well as being solutions to the Euler equations, these flows seemingly have the property that when $ABC \neq 0$, there are regions within the flow domain where particle trajectories are chaotic (Hénon 1966); neighbouring fluid particles diverge exponentially with time, so that there is a positive Liapunov exponent. This makes the flows potentially interesting both as a model for certain aspects of turbulence (Dombre *et al.* 1986; Moffatt 1986) and as a possible candidate for a fast kinematic dynamo whose growth rate remains finite as the electrical conductivity of the fluid tends to infinity (Childress 1979; Zeldovich, Ruzmaikin & Sokolov 1983; Arnol'd & Korkina 1983; Galloway & Frisch 1984, 1986; Moffatt & Proctor 1985).

Any ABC flow may also be considered as a steady solution of the Navier–Stokes equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{1.2}$$

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where p and ν respectively refer to the pressure and kinematic viscosity of the fluid, which is assumed without loss of generality to have unit density, and f is a driving force

$$f = \nu(A\sin z + C\cos y, B\sin x + A\cos z, C\sin y + B\cos x), \tag{1.3}$$

without which the flow would decay (note $e^{-\nu t}(u, v, w)$ is an exact solution of the Navier–Stokes equation when f is zero). At low Reynolds number $Re \equiv \nu^{-1}$ this steady solution is stable. It has been suggested by V. I. Arnol'd (private communication) that when the Reynolds number is increased, such flows may very rapidly become unstable and turbulent (in an Eulerian sense) since even in their unperturbed state they possess pre-existing chaotic particle paths. The latter represent a kind of 'Lagrangian turbulence', even though the flow itself is completely non-turbulent according to the normal interpretation of the word. This argument originally motivated the introduction of the ABC flows. A necessary first step in investigating such questions is to study the linear instability of these flows. Here we describe the results of this exercise for the restricted case where the velocity perturbations have the same scale l_0 as the basic flow, and we also describe some results for one or two of A, B, or C zero, which are valid somewhat more generally. This complements the results of Moffatt (1986), who showed using variational principles that the Euler (inviscid) flows are unstable to perturbations with a scale $l \gg l_0$.

The next section describes briefly the problem and the methods used to solve it; §3 gives the results of the computations, and §4 summarizes the conclusions.

2. The problem and method of solution

Let $\boldsymbol{v}$ be a perturbation with zero divergence to be added to the velocity $\boldsymbol{u}$ defined in (1.1). When this new total velocity is substituted into the Navier–Stokes equation, the resulting problem can be linearized to give the following equation for the evolution of the velocity perturbation:

$$\partial_t \boldsymbol{v} + \boldsymbol{u}\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} = \frac{-\nabla p}{\rho} + \frac{1}{Re}\nabla^2\boldsymbol{v}, \tag{2.1}$$

where p is the perturbed pressure and $1/Re$ is the viscosity in units based on the length of one side of the periodicity cube and the velocity amplitude of $\boldsymbol{u}$ (Re is thus a Reynolds number; for a consistent definition as A, B and C are varied, we normalize the values of A, B and C so that $A^2+B^2+C^2 = 3$). For comparison we write the electromagnetic induction equation describing the kinematic evolution of a magnetic field $\boldsymbol{B}$ subject to the action of the flow (1.1) in an electrically conducting fluid with scaled diffusivity R_m^{-1}:

$$\partial_t \boldsymbol{B} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} = \frac{1}{R_m}\nabla^2\boldsymbol{B}. \tag{2.2}$$

Equations (2.2) and (2.1) have some similarity, and (2.2) will henceforth be referred to as the corresponding magnetic problem; the analogy has been exploited in both directions by Moffatt (1985, 1986). The method developed for the magnetic case (Galloway & Frisch 1986) is easily extended to the hydrodynamic case. A Fourier representation is used for the velocity perturbation:

$$\boldsymbol{v}(x,y,z,t) = \sum_{l,\,m,\,n=-\frac{1}{2}N+1}^{\frac{1}{2}N} (v_x^{lmn}, v_y^{lmn}, v_z^{lmn})\, e^{i(lx+my+nz)}. \tag{2.3}$$

The pressure term in (2.1) can be eliminated with a projection operator; taking the divergence of (2.1) and expressing the result in Fourier space,

$$k^2 p^{lmn} = -k_i k_j (u_i * v_j + v_i * u_j)^{lmn}, \tag{2.4}$$

where the superscripts refer to Fourier mode $\boldsymbol{k} = (l, m, n), k = |\boldsymbol{k}|$, the suffices refer to Cartesian components, $*$ means a convolution product, and the summation convention is used. Substituting for p in (2.1) we find

$$\partial_t v_i^{lmn} = \mathrm{i} k_r \left(\frac{k_i k_s}{k^2} - \delta_{is} \right) Q_{rs}^{lmn} - \frac{k^2}{Re} v_i^{lmn}, \tag{2.5}$$

where $Q_{rs}^{lmn} = (u_r * v_s + v_r * u_s)^{lmn}$ is the lmnth Fourier mode extracted from the convolution sum. This turns out to be the same for the magnetic problem except that in the latter (i) the projection operator $(k_i k_s / k^2 - \delta_{is})$ is replaced by $-\delta_{is}$, and (ii) Q_{rs}^{lmn} becomes $(u_r * v_s - u_s * v_r)^{lmn}$. Modes with $k = 0$ are somewhat unphysical as they imply a net momentum for the perturbation in some direction, but they can be included if desired by conserving their amplitude during the evolution. Except for these two changes, the numerical scheme is constructed as described in Galloway & Frisch (1986), and the convolution term again has the particularly convenient form that was exploited there.

3. Results

For this paper we only investigated a few values for A, B and C, concentrating particularly on the case with $A = B = C$, which is the one most discussed in the literature. This is apparently unstable at high enough Re. The problem (2.1) is linear and admits growing solutions with time-dependence e^{st}; there is a discrete spectrum for the eigenvalues s, and if the calculation is started from random initial conditions, that eigenmode with largest Re(s) eventually predominates. In figure 1 the dependence of this largest Re(s) is shown as a function of Reynolds number. At high Re the growth rate seems to be asymptoting to a value around 0.216, comparable with the dynamical timescale, and short compared with the timescale Re for the diffusive decay of the unperturbed flow when not sustained by an imposed body force. Of course, we cannot be sure that this behaviour will persist at yet higher Re, but the numerical resolution available (up to 54^3) was felt insufficient to give reliable results beyond the values quoted here. (For the magnetic case we had access to a larger machine, and were able to resolve slightly more extreme cases.) The growing mode is oscillatory, with a period varying weakly from 21.6 at $Re = 15$ to 15.4 at $Re = 200$.

As described in the magnetic paper (Galloway & Frisch 1986), information on the detailed structure of the eigenfunction is extremely difficult to plot effectively. In the present case the kinetic-energy density v^2 is used as a diagnostic of the intensity of the eigenfunction (note that this is not the same as the perturbed kinetic-energy density, which contains in addition a term of lower order proportional to $\boldsymbol{u} \cdot \boldsymbol{v}$). Figure 2 shows a stereoscopic plot of this quantity for $Re = 250$, and figure 3 illustrates contour levels for eight sections $z = \text{constant}$ through the periodicity cube, for $Re = 100$. Superimposed on figure 3 are Poincaré plots showing successive crossings of one fluid particle carried by the unperturbed flow as it intersects each section, the planes being identified modulo 2π as the particle leaves the periodicity cube (see Dombre *et al.* 1986 for a full description). The particle is chosen to be one whose trajectory traces out a chaotic region. It can be seen that as in the magnetic problem

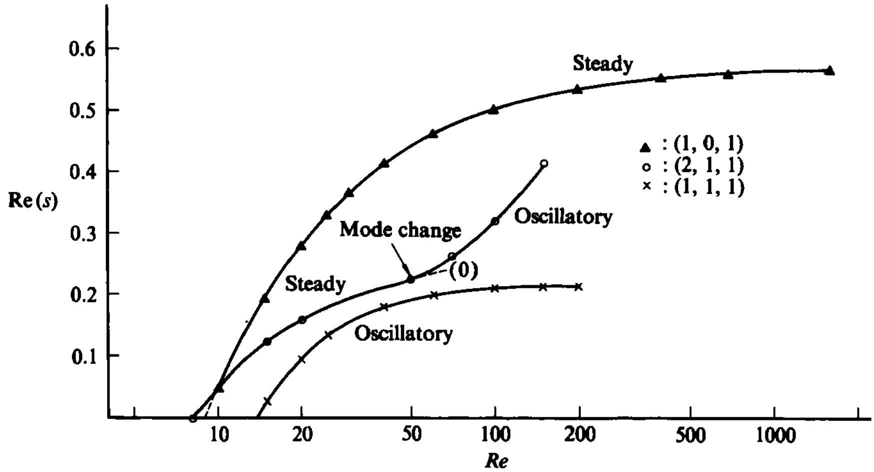

FIGURE 1. Value of growth rate $Re(s)$ for fastest growing mode as a function of Reynolds number: (*a*) for $A = B = C$; (*b*) for $A:B:C = 2:1:1$; (*c*) for $B = 0$, $A = C$.

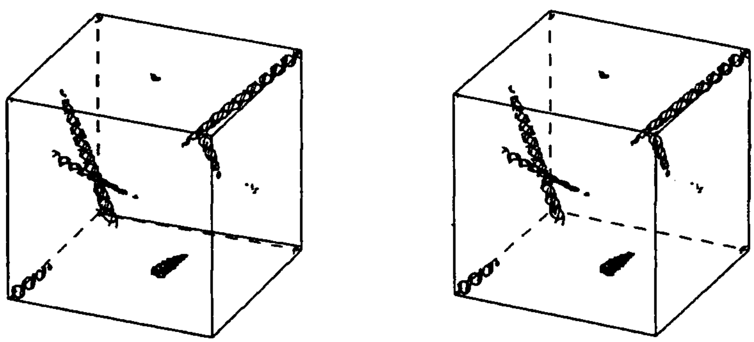

FIGURE 2. Stereoscopic plot of the kinetic-energy density v^2 of disturbance for Reynolds number 250, $A = B = C$.

the intensity is related to the regions of chaos, and is concentrated in cigar-shaped features whose origin is probably due to advection and straining in the vicinity of stagnation points. In the present problem, these features are somewhat more spread out; this is also apparent from the vorticity spectra (not shown here), which peak at a wavenumber of order 1, compared with a current spectrum peak at order $R_m^{\frac{1}{2}}$ for the MHD problem. A similar difference in the intermittency of velocity fields and magnetic fields is also found in MHD turbulence simulations (Meneguzzi, Frisch & Pouquet 1981).

We also examined the solution for possible symmetries associated with the group theoretical symmetries of the basic flow (see the discussion and references in Galloway & Frisch 1986). In the magnetic case, solutions both with and without symmetry breaking were found; in the present instance there is only one branch of growing solutions, and no evidence of any symmetry was found, even though in this respect the same possibilities exist for the two problems.

Other values of A, B and C have been tried, though it is hard to know on what basis to do this systematically. The flow has stagnation points if and only if A^2, B^2 and C^2 can form a triangle ($C^2 \leqslant A^2+B^2$, say). Thus we tried $A:B:C = 2:1:1$ (as explained in §2, A, B and C were normalized so that $A^2+B^2+C^2 = 3$), to see whether a lack of stagnation points made any difference. The flow is still unstable, though

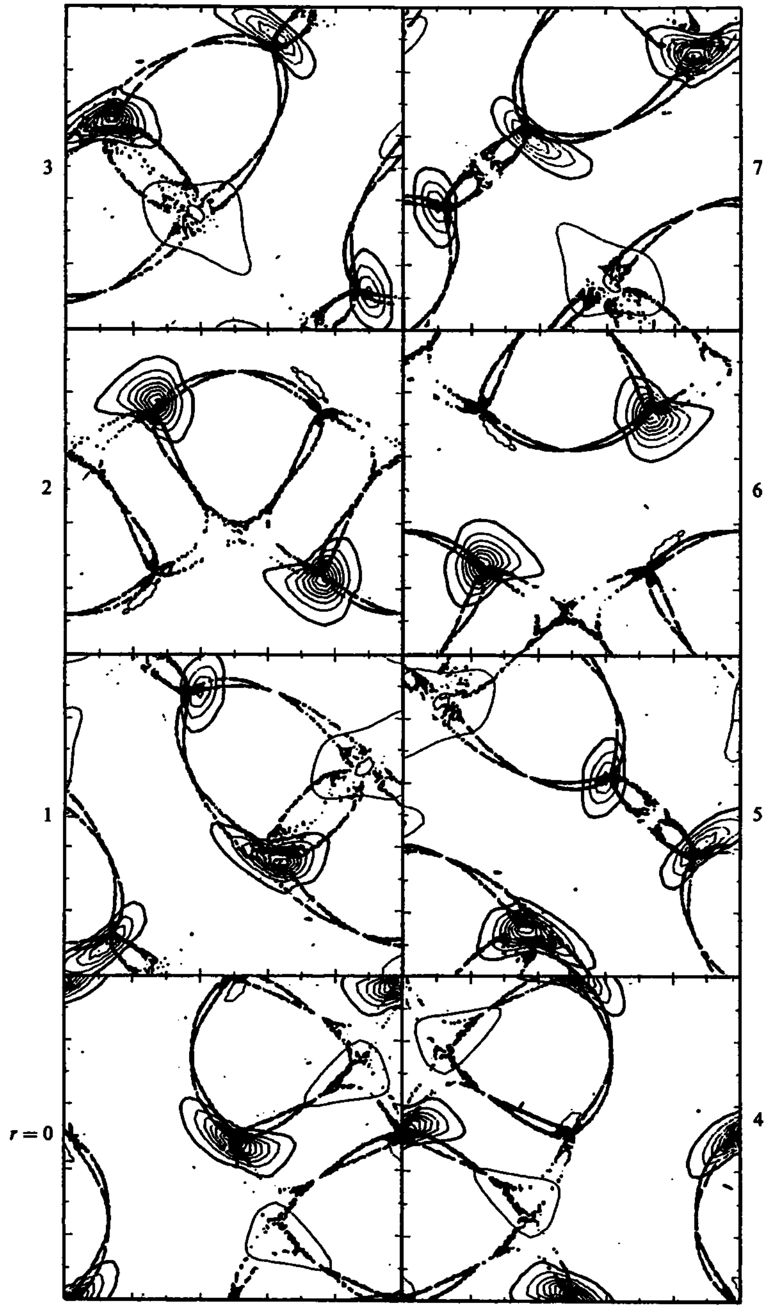

FIGURE 3. Poincaré sections of particle trajectories crossing the planes $z = r\pi/4 (r = 0, 1, \ldots 7)$. One particle only is tracked; its successive intersections suffice to delineate a connected region where the flow is chaotic. Superimposed are contours of intensity of v^2 in the same planes, normalized so that the maximum contour is derived over the whole cube. Note the clear relation between the strong v features and the chaos. ($A = B = C = 1$; the v-field is for $Re = 100$).

in this case at lower Re ($\leqslant 50$) the most unstable mode has purely exponential growth, i.e. the eigenvalue is real. As Re is increased, this mode gives way to an oscillatory mode which grows faster; it is not possible to follow this mode for very long before running out of resolution, but at least until then the growth rate is still increasing rapidly with Re.

When one of A, B or C is zero the basic flow is integrable and there are no regions where streamlines are chaotic. As in the magnetic case, if say the parameter B associated with x-variability is set to zero, each perturbed mode of the form $\boldsymbol{u} = \mathrm{e}^{ilx}\boldsymbol{u}^{\dagger}(y,z)$ evolves independently and the time-dependence of $\boldsymbol{u}^{\dagger}$ can be followed with a two-dimensional code. Thus much higher Re can be attained. Shown on figure 1 is the growth rate obtained using such a code for the case $B = 0, A = C, l = 2$. This is not necessarily the most unstable l, but it can be seen that the behaviour is qualitatively similar to the case $A = B = C$, with an oscillatory growing mode. In particular, the instability remains 'fast' as Re becomes large, as is often the case in hydrodynamic stability problems. In the magnetic case, if attention is similarly fixed on a given x-wavenumber l, the growth rate falls off as a negative power of R_m. However, Soward (1987) has studied this problem for the case where the wavelength is *short* compared with that of the undisturbed flow, and finds that the most unstable l scales as $R_m^{\frac{1}{2}}$, with a growth rate falling off very slowly as $\log(\log R_m)/\log R_m$. Analogous results may exist for the problem investigated in this paper, though it is not clear what kind of correspondence should really be expected between the two cases. It would appear worthwhile to extend our numerical methods to include an arbitrary Floquet multiplier in order to treat scale separation more systematically and allow the wavenumber to vary continuously.

When *two* of A, B and C are zero, our code yields stable solutions; in this form the problem is similar to one solved analytically by Meshalkin & Sinai (1961) and Green (1974), and in fact it can be shown that the linear stability properties are identical with those of the latter problem. (A demonstration of this is given in the Appendix.) In particular, there is only instability for perturbations with wavelengths longer than that of the basic flow. These are not included in our numerical solutions, and thus no instability is manifest in the results.

The difference in the behaviour between the magnetic and stability problems in the integrable case shows, perhaps not surprisingly, that despite the similarity in form of the governing equations the nature of the solutions can be quite different. There are also similarities; in the case with $A = B = C$ intense structures accumulate in the same location near stagnation points. As Moffatt (1985, 1986) has stressed, for zero diffusivity, although the equilibrium solutions in the two cases are analogous, the perturbation problems are different because in the former, the field lines are carried with the fluid, whereas in the latter the vorticity, which is the curl of the analogous variable, is the quantity convected.

4. Conclusion

We have seen that in almost all cases studied, the ABC flows are unstable to perturbations with the same periodicity as the basic flow, provided that a critical Reynolds number of the order of ten is exceeded. For small Reynolds number, these modes are stable, but a multiple-scale analysis, not given here, shows that a low Re there is still instability provided that perturbations with wavelength much greater than the scale of the basic flow are included. The same holds for the case where two of A, B or C vanish (see the Appendix); in this case modes with the same scale as the basic flow are stable for all finite Re, but longer-wavelength modes become unstable for $Re > \sqrt{2}$. Moffatt (1986) has investigated the strictly inviscid case, and has shown using variational methods that for general A, B, C there are always unstable modes with a scale much larger than the scale of the basic flow; for these modes he finds a growth time proportional to $k^{-\frac{3}{2}}$, where k is the (small) wavenumber of the perturbation. It seems likely that since for high enough Re the modes found numerically in this paper evolve on the turnover timescale, they are in fact more unstable than those treated by Moffatt. In any case, the often-met conjecture that Beltrami flows are particularly stable structures is not supported by the results for this family of flows.

For the future, it will be interesting to find the nature of the bifurcated solutions of the Navier–Stokes equations (2.2) once the basic ABC flow has become unstable; it will then be possible to test Arnol'd's conjecture that the underlying chaotic nature of the basic flows renders them liable to an extremely rapid development of Eulerian turbulence. The methods used in this paper cannot handle this problem because of their restriction to one wavenumber for the unperturbed flow; however, using a different scheme (e.g. a pseudospectral method), the solution should be straightforward.

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Appendix. The case where two of A, B, C vanish

Meshalkin & Sinai (1961) and Green (1974) both studied the stability of the two-dimensional 'Kolmogorov' flow $\boldsymbol{u} = (0, \sin x)$. They found that it is stable for any finite Reynolds number against perturbations with the same 2π-periodicity in x and y as the basic flow. Above a critical value of $\sqrt{2}$ for the Reynolds number the flow is unstable against perturbations of very long wavelength. We shall now show that the linear stability problem for three-dimensional flows such as $\boldsymbol{u} = (0, \sin x, \cos x)$, obtained by putting two of A, B or C zero in (1.1), is essentially the same.

First, let us consider perturbations depending only on x and y. We denote by $\boldsymbol{u}_{\mathrm{H}} = (0, \sin x)$ and $\boldsymbol{v}_{\mathrm{H}} = (v_x, v_y)$ the projections of the basic flow and the perturbation on the (x, y)-plane, and by $u_z = \cos x$ and v_z their z-components. Let $\boldsymbol{\nabla}_{\mathrm{H}}$ be the (x, y)-gradient operator. The basic stability equation (2.1) can now be split into the pair

$$\partial_t \boldsymbol{v}_{\mathrm{H}} + \boldsymbol{u}_{\mathrm{H}} \cdot \boldsymbol{\nabla}_{\mathrm{H}} \boldsymbol{v}_{\mathrm{H}} + \boldsymbol{v}_{\mathrm{H}} \cdot \boldsymbol{\nabla}_{\mathrm{H}} \boldsymbol{u}_{\mathrm{H}} = -\boldsymbol{\nabla}_{\mathrm{H}} p + \nu \nabla^2 \boldsymbol{v}_{\mathrm{H}}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{v}_{\mathrm{H}} = 0, \tag{A 1}$$

$$\partial_t v_z + \boldsymbol{u}_{\mathrm{H}} \cdot \boldsymbol{\nabla}_{\mathrm{H}} v_z - v_x \sin x = \nu \nabla^2 v_z. \tag{A 2}$$

We observe that (A 1) is precisely the equation governing the two-dimensional perturbations of the Kolmogorov flow. The z-component v_z satisfies an advection equation with a source term; hence it cannot produce additional instabilities beyond those of the Kolmogorov flow.

Next, we consider perturbations with an arbitrary x-, y- and z-dependence, and suppose that the y, z part of this dependence is proportional to $e^{i(my+nz)}$. We set

$$\frac{m}{(m^2+n^2)^{\frac{1}{2}}} = \cos\theta, \quad \frac{n}{(m^2+n^2)^{\frac{1}{2}}} = \sin\theta, \tag{A 3}$$

and rotate the coordinate system by an angle θ around the x-axis, so that the new Y-axis is in the former $(0, m, n)$-direction. The basic flow is now $(0, \sin(x+\theta), \cos(x+\theta))$ and the perturbation is only Y-dependent. A change of phase is irrelevant for the stability of the Kolmogorov flow; thus we are back to the former case, which concludes the proof.

Finally, we mention that the stability of the helical flow $(0, \cos(x/L), -\sin(x/L))$, obviously equivalent to the case here studied, was investigated recently by Bayly & Yakhot (1986). Our derivation is more compact in so far as the problem is reduced to the one already investigated by Meshalkin & Sinai (1961). On the other hand, Bayly & Yakhot's field-theoretical method gives a rather direct derivation of the critical value $\sqrt{2}$.

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